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TRANSFORMATIONS ON [0, 1] WITH INFINITE INVARIANT MEASURES

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ABSTRACT

Under certain regularity conditions a real transformation with indifferent fixed points has an infinite invariant measure equivalent to Lebesgue measure. In this paper several ergodic properties of such transformations are established.

Introduction

In [20] we studied the invariant densities of real transformations with indifferent fixed points. The purpose of the present paper is to give a more detailed analysis of the ergodic behaviour of such transformations using the density estimates obtained in [20] and the fact that the associated jump transformations satisfy Rényi's condition. Like null recurrent Markov chains or inner functions of the upper half plane, transformations of this type are good examples to illustrate the statistical laws governing the iteration of maps preserving infinite measures.

Section 1 contains the necessary definitions and notations as well as some general remarks on auxiliary transformations which are essential tools in our investigation. In section 2 we prove exactness and rational ergodicity. In section 3 we show that for a given transformation T the class of sets with the same minimal wandering rate is large enough to provide an isomorphism invariant. We also show how to calculate the minimal wandering rate in case T admits expansions at the indifferent fixed points. In section 4 we deal with the problem of calculating the entropy of the transformations in question. At the same time we derive a criterion to decide whether it is finite or infinite, and an analogue to the Theorem of McMillan.

1. Preliminaries

We begin with a brief survey of those properties of induced and jump transformations which are of interest here using a slightly more general concept.

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Let $(X,\mathscr{R})$ be a measurable space and $T:X\to X$ a measurable transformation. Let $A\in\mathscr{R}$, and $n:A\to\mathbf{N}$ be a measurable map such that $T^{n(x)}(x)\in A$ for every $x\in A$. Then we can define a transformation $T_{A,n}:A\to A$ by $T_{A,n}(x)=T^{n(x)}(x)$. Putting

$$B_k=\{x\in A:n(x)=k\}\qquad (k\geqq 1)$$

we have $T_{A,n}^{-1}(E)=\bigcup_{k=1}^{\infty}(B_k\cap T^{-k}E)$ for each $E\subseteq A$. This shows $T_{A,n}$ is measurable with respect to $A\cap\mathscr{R}$. Recall the following well-known special cases.

(i) Let $A\subseteq\bigcup_{n=1}^{\infty}T^{-n}A$. Then

$$n(x)=\min\{n\geqq 1:T^n(x)\in A\},\qquad x\in A,$$

defines the induced transformation T_A on A (cf. [9]).

(ii) Let α_1 be an at most countable measurable partition of X; let α_n be the set of atoms of $\bigvee_{i=0}^{n-1}T^{-i}\alpha_1$ and β an arbitrary subset of $\bigcup_{n=1}^{\infty}\alpha_n$ with $\bigcup_{Z\in\beta}Z=X$. Then the transformation defined by

$$n(x)=\min\{n\geqq 1:x\in Z\in\alpha_n\cap\beta\},\qquad x\in X,$$

is known as the jump transformation over β (cf. [16]).

(iii) Let $B\in\mathscr{R}$, $\bigcup_{n=0}^{\infty}T^{-n}B=X$, and

$$n(x)=1+\min\{n\geqq 0:T^n(x)\in B\},\qquad x\in X.$$

The transformation obtained in this way may be called the 'first passage map' with respect to B. It is closely related to T_B (cf. [17]).

Now let $T_{A,n}$ be defined as above. Taking into account that $B_k\cap T^{-k}(E\cap A)=B_k\cap T^{-k}E$ it is easily seen that the set $E\cap A$ is invariant for $T_{A,n}$ whenever E is invariant for T. Thus, if σ is a measure on $\mathscr{R}$ such that T is non-singular with respect to σ and $X=\bigcup_{n=0}^{\infty}T^{-n}A$ (mod 0), we have:

(1.1) If $T_{A,n}$ is ergodic with respect to $\sigma|_{A\cap\mathscr{R}}$, then T is ergodic with respect to σ.

From $T_{A,n}^{-s}(E)\subseteq\bigcup_{k=s}^{\infty}T^{-k}E$, $E\subseteq A$, $s\geqq 1$, it follows that $W\cap A$ is wandering for $T_{A,n}$, if W is wandering for T. Hence given a measure σ on $\mathscr{R}$ such that $A=X$ (mod 0) we have:

(1.2) If $T_{A,n}$ is conservative with respect to $\sigma|_{A\cap\mathscr{R}}$, then T is conservative with respect to σ.

Finally, every invariant measure for $T_{A,n}$ (A, n arbitrary) yields an invariant measure for T. To prove this, let

$$D_k = \{x \in A : n(x) \geqq k\} \qquad (k \geqq 1)$$

and ν be a measure on $A \cap \mathcal{R}$. For $s = -1,0,1,2,\cdots$, define the measures ν_s by the formula

$$\nu_s = \sum_{k=1}^{\infty} \nu(T^{-s-k}E \cap D_k), \qquad E \in \mathcal{R}.$$

Then,

$$\text{if } \nu \text{ is invariant for } T_{A,n},\ \nu_s \text{ is invariant for } T, \tag{1.3}$$
$$\text{i.e. } \nu_s = \nu_t \text{ for all } s,t \in \{-1,0,1,2,\cdots\}.$$

This very useful formula has a very short proof. For, using $D_k = D_{k+1} \cup B_k$ ($k \geqq 1$), we have

$$\nu_s(T^{-1}E) = \sum_{k=2}^{\infty} \nu(T^{-s-k}E \cap D_k) + \nu(T_{A,n}^{-1}(T^{-s-1}E \cap A)) = \nu_s(E)$$

(cf. [13] for induced transformations ($s = 0$), [16] for jump transformations ($s = -1$)).

Here we shall be concerned with a class of transformations $T:[0,1] \to [0,1]$ specified below. In fact, these transformations as well as the auxiliary transformations occurring in this paper will generally be defined only up to sets of Lebesgue measure zero. Henceforth we shall not mention this explicitly.

Let $\zeta_1 = \{B(i): i \in I\}$ be a collection of disjoint subintervals of $[0,1]$, $|I| \geqq 2$, with $\lambda([0,1] \setminus \bigcup_{i \in I} B(i)) = 0$, where λ denotes the Lebesgue measure on the σ-field $\mathcal{R}$ of Lebesgue measurable subsets of $[0,1]$. Then we assume:

(1) $T_{|B(i)}$ is twice differentiable, and $\overline{TB(i)} = [0,1]$ for all $i \in I$. Every $B(i)$ contains exactly one fixed point x_i, and the set $J = \{i \in I : T'(x_i) = 1\}$ is finite.

(2) $T'(x) \geqq \rho(\varepsilon) > 1$ for all $x \in \bigcup_{i \in I} B(i) \setminus \bigcup_{j \in J} (x_j - \varepsilon, x_j + \varepsilon)$, $\forall \varepsilon > 0$.

(3) For $j \in J$, T' is decreasing on $(x_j - \eta, x_j) \cap B(j)$ and increasing on $(x_j, x_j + \eta) \cap B(j)$ for some $\eta > 0$.

(4) $|T''(x)|\, T'(x)^{-2}$ is bounded on $\bigcup_{i \in I} B(i)$.

Let $\mathcal{T}$ denote the class of all such transformations and $\mathcal{T}_R$ the subclass of those among them for which $J = \varnothing$. As is well known $T \in \mathcal{T}$ belongs to $\mathcal{T}_R$, if and only if T satisfies Rényi's condition (cf. [14]). In particular, every $T \in \mathcal{T}_R$ has a finite ergodic invariant measure equivalent to λ (see also Lemma 3 in section 3). Furthermore, it is easy to verify that ζ_1 is a generator for every $T \in \mathcal{T}$. We shall use the following notations:

$$B(k_1, \cdots, k_n) = \bigcap_{i=1}^{n} T^{-i+1} B(k_i), \qquad (k_1, \cdots, k_n) \in I^n, \quad n \geqq 1;$$

$$B_n(i) = B(\underbrace{i, \cdots, i}_{n \text{ times}}), \quad i \in I, \qquad \mathcal{D}_n = \{B_n(j) : j \in J\},$$

$$\zeta_n = \{B(k_1, \cdots, k_n) : (k_1, \cdots, k_n) \in I^n\}, \qquad \zeta = \bigcup_{n=1}^{\infty} \zeta_n.$$

As in [20], let T^* denote the jump transformation over the cylinder class $\beta = \zeta \setminus \bigcup_{n=1}^{\infty} \mathcal{D}_n$ (cf. (ii)). With the notations introduced above we have in this case

$$D_n = \bigcup_{Z \in \mathcal{D}_{n-1}} Z, \quad n \geqq 2, \qquad B_1 = \bigcup_{k \in I \setminus J} B(k),$$

$$B_n = \bigcup_{j \in J} \bigcup_{k \neq j} B(\underbrace{j, \cdots, j}_{n-1}, k), \qquad n \geqq 2.$$

Let $I^* = \{(k_1, \cdots, k_n) : B(k_1, \cdots, k_n) \subseteq B_n, n \geqq 1\}$. By theorem 2 and corollary 2 in [20], $T^* \in \mathcal{T}_R$. In particular,

(1.4) there exists a constant C such that

$$\operatorname*{ess\,sup}_{x \in [0,1]} \omega(k_1, \cdots, k_n)(x) \leqq C \operatorname*{ess\,inf}_{x \in [0,1]} \omega(k_1, \cdots, k_n)(x)$$

for all $(k_1, \cdots, k_n) \in I^n$ $(n \geqq 1)$ of the form $(k_1^*, \cdots, k_t^*)$ with $k_i^* \in I^*$, $1 \leqq i \leqq t$, where

$$\omega(k_1, \cdots, k_n)(x) = \frac{d}{dx} f_{k_1, \cdots, k_n}(x)$$

and $f_{k_1, \cdots, k_n}$ is the inverse of T^n restricted to $B(k_1, \cdots, k_n)$.

According to (1.3) $T \in \mathcal{T}$ has an invariant measure $\mu \sim \lambda$. As a consequence of (1.1) and (1.2),

(1.5) T is conservative and ergodic with respect to $\mu(\lambda)$.

It was shown in [20] that $d\mu/d\lambda$ satisfies

$$c_1 \prod_{j \in J} G_j(x) \leqq \frac{d\mu}{d\lambda}(x) \leqq c_2 \prod_{j \in J} F_j(x) \quad \text{a.e.} \quad (c_1, c_2 > 0),$$

where

$$\left.\begin{aligned} G_j(x) &= (x - x_j)(Tx - x)^{-1} \\ F_j(x) &= (x - x_j)(x - f_j(x))^{-1} \end{aligned}\right\} \quad \text{for } x \in B(j), x \neq x_j,$$

$$G_j(x) = F_j(x) = 1 \qquad \text{for } x \in [0,1] \setminus B(j).$$

From $(Tx - x)(x - f_j(x))^{-1} = T'(\xi_x)$ (ξ_x between x and $f_j(x)$) for $x \in B(j)$ it follows that

$$\lim_{x \to x_j} F_j(x)/G_j(x) = 1 \qquad \text{for all } j \in J.$$

Using this and a continuity argument one sees that $F_j(x)/G_j(x)$ is bounded on $B(j)$ and hence on $[0,1]$. Therefore the above estimates can be written in the following more compact form:

$$\begin{aligned} \frac{d\mu}{d\lambda}(x) &= h_\mu(x) \prod_{j \in J} G_j(x) \\ &= \tilde{h}_\mu(x) \prod_{j \in J} F_j(x), \qquad 0 < c_1 \leqq h_\mu, \tilde{h}_\mu \leqq c_2 < \infty. \end{aligned} \tag{1.6}$$

If $j \in J$, then by condition (4)

$$|Tx - x| = \tfrac{1}{2}|T''(\xi_x)|(x - x_j)^2 \leqq \text{const.}(x - x_j)^2$$

in a neighbourhood of x_j. This shows that the invariant measure μ is infinite for $T \in \mathcal{T} \setminus \mathcal{T}_R$. Some examples belonging to the class $\mathcal{T} \setminus \mathcal{T}_R$ can be found in [2], [4], [5], [10], [12], [13], [18], [19], [20].

REMARKS. (i) If T' is replaced by $|T'|$ in the assumptions for T, (1.6) remains true as long as T is increasing on every interval $B(j)$, $j \in J$. However, if T is decreasing on $B(j)$, G_j has to be replaced by $\hat{G}_j$, where

$$\hat{G}_j(x) = (x - x_j)(T^2 x - x)^{-1}, \quad x \in B(j,j), \quad \hat{G}_j(x) = 1 \text{ elsewhere.}$$

The second expression has to be modified analogously. All other statements also hold when T is decreasing on $B(j)$ for some or all $j \in J$.

(ii) As mentioned before, the conservativity of $T \in \mathcal{T} \setminus \mathcal{T}_R$ follows via (1.2) from the conservativity of T^*. By the following argument it may be possible to prove conservativity by a short calculation, provided the invariant measure is known.

$$\text{If} \quad \sum_{j\in J}\sum_{k=1}^{\infty}(f_j^k)'(x) \geqq c\frac{d\mu}{d\lambda}(x) \quad \lambda\text{-a.e.} \quad (c>0), \tag{1.7}$$
$$\text{then } T \text{ is conservative with respect to } \lambda.$$

PROOF. Let $W \in \mathscr{R}$ be wandering with respect to T. Then $\sum_{k=1}^{\infty}\lambda(T^{-k}W) \leqq 1$, hence

$$\lim_{n\to\infty} r_n = 0 \qquad \text{for } r_n = \sum_{k=n+1}^{\infty}\lambda(T^{-k}W).$$

Now,

$$\begin{aligned} r_n &= \sum_{k=1}^{\infty}\lambda(T^{-k}(T^{-n}W)) \geqq \sum_{k=1}^{\infty}\lambda\Big(T^{-k}(T^{-n}W)\cap \bigcup_{j\in J} B_k(j)\Big) \\ &= \sum_{j\in J}\sum_{k=1}^{\infty}\lambda(T^{-k}(T^{-n}W)\cap B_k(j)) \\ &= \int_{T^{-n}W}\Big(\sum_{j\in J}\sum_{k=1}^{\infty}(f_j^k)'(x)\Big)d\lambda(x) \\ &\geqq c\int_{T^{-n}W}\frac{d\mu}{d\lambda}(x)\,d\lambda(x) \\ &= c\mu(W) \qquad \text{for all } n \geqq 1. \end{aligned}$$

Thus, $\mu(W) = \lambda(W) = 0$. □

The following well-known examples illustrate the application of (1.7).

EXAMPLE 1.

$$T(x) = x/(1-x) \pmod 1, \qquad \frac{d\mu}{d\lambda}(x) = x^{-1},$$

$$B(i) = [i/(i+1),(i+1)/(i+2)), \qquad i = 0,1,2,\cdots, \quad J=\{0\},$$

$$(f_0^k)'(x) = (1+kx)^{-2},$$

$$\sum_{k=1}^{\infty}(f_0^k)'(x) \geqq x^{-1}\sum_{k=1}^{\infty}(1/(1+kx) - 1/(1+(k+1)x)) \geqq \tfrac{1}{2}x^{-1}.$$

EXAMPLE 2.

$$T(x) = \tan x \pmod \pi \quad \text{on } (-\pi/2,\pi/2), \qquad \frac{d\mu}{d\lambda}(x) = \sin^{-2}x,$$

$$B(i) = (\arctan(2i-1)\pi/2, \arctan(2i+1)\pi/2), \qquad i\in\mathbf{Z}, \quad J=\{0\},$$

$$f_0'(x) = (1+x^2)^{-2},$$

$$\sum_{k=1}^{\infty}(f_0^k)'(x) \geqq \sum_{k=1}^{\infty}(f_0'(x))^k = x^{-2} \geqq (4/\pi^2)\sin^{-2}x.$$

2. Exactness and rational ergodicity

It is well known that the transformations $T \in \mathcal{T}_R$ are exact endomorphisms, i.e.

$$\bigcap_{n=1}^{\infty} T^{-n}\mathcal{R} = \{\phi, [0,1]\} \ (\text{mod } 0) \qquad \text{(cf. [15])}.$$

As the condition $\overline{TB(i)} = [0,1]$ $(i \in I)$ suggests, this is true for all $T \in \mathcal{T}$.

THEOREM 1. *Every $T \in \mathcal{T}$ is an exact endomorphism.*

PROOF. Let $\zeta_n^* = \{B(k_1^*, \cdots, k_n^*): k_i^* \in I^*,\ 1 \leqq i \leqq n\}$ be the set of T^*-cylinders of rank n. Define the functions $n_k(x)$, $k \geqq 1$, by

$$n_k(x) = j \Leftrightarrow x \in Z \in \zeta_k^* \cap \zeta_j.$$

Then n_k is defined a.e. on $[0,1]$ and fulfils

$$(T^*)^k(x) = T^{n_k(x)}(x).$$

Let $A \in \bigcap_{n=1}^{\infty} T^{-n}\mathcal{R}$, i.e. for every n there exists a set $A_n \in \mathcal{R}$ such that $A = T^{-n}A_n$, and let $\lambda(A) > 0$. If $Z \in \zeta_k^*$ and $x \in Z$, then

$$E(1_A \,\|\, \zeta_k^*)(x) = \lambda(A \cap Z)/\lambda(Z) = \lambda((T^*)^{-k}(A_{n_k(x)}) \cap Z)/\lambda(Z).$$

In view of (1.4) this implies

$$C^{-1}\lambda(A_{n_k(x)}) \leqq E(1_A \,\|\, \zeta_k^*)(x) \leqq C\,\lambda(A_{n_k(x)}). \tag{2.1}$$

Let

$$\delta(x) = \liminf_{k \to \infty} \lambda(A_{n_k(x)}).$$

Since $\bigvee_{k=1}^{\infty} \zeta_k^* = \mathcal{R}$ (mod 0) it follows from the left hand side of (2.1) and the martingale theorem that

$$1_A(x) \geqq C^{-1}\delta(x) \quad \text{a.e.}$$

The assertion of the theorem is proved if we show $\delta(x) > 0$ a.e. To do this, let

$$K = \left\{ x \in [0,1] : \lim_{k \to \infty} E(1_A \,\|\, \zeta_k^*)(x) = 1_A(x) \right\}.$$

Then $\lambda(A \cap K) = \lambda(A) > 0$ and $\bigcup_{k=0}^{\infty} (T^*)^{-k}(A \cap K) = [0,1]$ (mod 0) since T^* is conservative and ergodic with respect to λ. Now let $y = (T^*)^t(x) \in A \cap K$. The right hand side of (2.1) then yields

$$1 = 1_A(y) = \lim_{k\to\infty} E(1_A \| \zeta_k^*)(y) \leqq C \liminf_{k\to\infty} \lambda(A_{n_k(y)}).$$

Hence $\lambda(A_{n_k(y)}) \geqq \varepsilon(y) = \varepsilon > 0$ for all $k \geqq 1$. In particular, $\delta(x) > 0$ if $t = 0$. For $t \geqq 1$ we argue as follows. Because of

$$n_k(y) = n_{k+t}(x) - n_t(x), \qquad k \geqq 1,$$

and

$$A_n = T^k A_{n-k} \text{ (mod 0)}, \qquad 0 \leqq k < n, \quad n \geqq 1,$$

we have

$$A_{n_{k+t}(x)} = T^{n_t(x)}(x)(A_{n_k(y)}) \qquad \text{for every } k \geqq 1.$$

Let I_0 be a finite subset of I such that $\Sigma_{i\in I_0}\lambda(B(i)) > 1 - \varepsilon/2$. Then

$$\max_{i\in I_0} \lambda(A_{n_k(y)} \cap B(i)) \geqq \varepsilon/2|I_0|.$$

Thus,

$$\begin{aligned}\lambda(T^{n_t(x)}(A_{n_k(y)})) &\geqq \max_{i\in I_0} \lambda(T^{n_t(x)}(A_{n_k(y)} \cap B(i))) \\ &\geqq \max_{i\in I_0} \lambda(A_{n_k(y)} \cap B(i)) \\ &\geqq \varepsilon/2|I_0|.\end{aligned}$$

Therefore

$$\lambda(A_{n_k(x)}) \geqq \varepsilon/2|I_0| \qquad \text{for } k > t.$$

This completes the proof. □

In [1] a conservative ergodic measure preserving transformation on a σ-finite measure space $(X, \mathcal{R}, \mu)$ is called rationally ergodic, if a set $A \in \mathcal{R}$ of positive finite measure exists such that

$$\sup_{n\geqq 1} \int_A \left(\sum_{k=0}^{n-1} 1_A \circ T^k / a_n(A)\right)^2 d\mu < \infty, \tag{2.2}$$

where $a_n(A) = \Sigma_{k=0}^{n-1}\mu(A \cap T^{-k}A)$. This condition implies that the sequence $\{(1/a_n(A))\Sigma_{k=0}^{n-1} 1_A \circ T^k : n \geqq 1\}$ is uniformly integrable on A or, equivalently, that the following ratio limit theorem holds for all A_1, A_2, C_1, $C_2 \in A \cap \mathcal{R}$ of positive measure:

(2.3) $$\lim_{n\to\infty} a_n(A_1, C_1)/a_n(A_2, C_2) = \mu(C_1)/\mu(C_2),$$

where $a_n(A_i, C_i) = (1/\mu(A_i))\sum_{k=0}^{n-1}\mu(A_i \cap T^{-k}C_i)$ $(i = 1, 2)$ (cf. [1], [7]).

For an intuitive interpretation of (2.3) note that $a_n(A_i, C_i)$ is the expected number of visits to the set C_i before time n when the process starts in A_i.

We shall show that in our case, (2.2), and hence (2.3), is valid for all measurable sets of positive measure which are bounded away from the fixed points x_j, $j \in J$. Let $T \in \mathcal{T}$ and

$$B(T) = \{A \in \mathcal{R} : 0 < \mu(A) < \infty, A \text{ satisfies (2.2)}\}.$$

Then we have

THEOREM 2. *$[0,1]\setminus\bigcup_{j\in J}(x_j - \varepsilon, x_j + \varepsilon) \in B(T)$ for every $\varepsilon > 0$.*

PROOF. For $T \in \mathcal{T}_R$ the assertion is obvious. Hence assume $J \neq \emptyset$. Take $n \geqq 1$ with $D_{n+1} \subseteq \bigcup_{j\in J}(x_j - \varepsilon, x_j + \varepsilon)$ and put $A = \bigcup_{k=1}^{n} B_k$. Since $A \in B(T)$ implies $A' \in B(T)$ for every measurable subset A' of A with positive measure it suffices to prove $A \in B(T)$. As can be seen from (1.6), there are constants $d_2 \geqq d_1 > 0$ such that

$$d_1 \leqq \frac{d\mu}{d\lambda} \leqq d_2 \qquad \text{a.e. on } A.$$

Therefore,

$$\int_A \left(\sum_{k=0}^{n-1} 1_A \circ T^k\right)^2 d\mu = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \mu(A \cap T^{-i}A \cap T^{-j}A)$$

$$\leqq 2d_2 \sum_{i=0}^{n-1}\sum_{j=i}^{n-1} \lambda(A \cap T^{-i}A \cap T^{-j}A).$$

Let $i \geqq 0$ be fixed and

$$\alpha = \{\mathbf{k}_{i+n} = (k_1, \cdots, k_{i+n}) \in I^{i+n} : B(\mathbf{k}_{i+n}) \subseteq A \cap T^{-i}A\}.$$

Since $|T''(x)|\,T'(x)^{-2}$ is bounded there is a constant $M = M(n)$ such that

(2.4) $$\operatorname*{ess\,sup}_{x\in[0,1]} \omega(\mathbf{k}_t)(x) \leqq M \operatorname*{ess\,inf}_{x\in[0,1]} \omega(\mathbf{k}_t)(x)$$

for all $\mathbf{k}_t \in I^t$, $1 \leqq t \leqq n$.

If $\mathbf{k}_{i+n} \in \alpha$, then either $k_{i+1} \in I\setminus J$ or $(k_{i+1}, \cdots, k_{i+s}) = (j, \cdots, j, k)$, $j \in J$, $k \neq j$, for some $s \in \{2, \cdots, n\}$. Hence there exists an index t, $i + 1 \leqq t \leqq i + n$, such that

$$(k_1, \cdots, k_t) = (k_1^*, \cdots, k_r^*), \qquad k_i \in I^*, \quad 1 \leqq i \leqq r.$$

Taking into account that

$$\omega(\boldsymbol{k}_{i+n})(x)=\omega(k_1,\cdots,k_i)(f_{k_{i+1},\cdots,k_{i+n}}(x))\,\omega(k_{i+1},\cdots,k_{i+n})(x)$$

we get from (1.4) and (2.4)

$$\operatorname*{ess\,sup}_{x\in[0,1]}\omega(\boldsymbol{k}_{i+n})(x)\leqq CM\operatorname*{ess\,inf}_{x\in[0,1]}\omega(\boldsymbol{k}_{i+n})(x).$$

Therefore we obtain for $j>i+n$:

$$\begin{aligned}\lambda(A\cap T^{-i}A\cap T^{-j}A)&=\sum_{\boldsymbol{k}_{i+n}\in\alpha}\lambda(B(\boldsymbol{k}_{i+n})\cap T^{-(i+n)}(T^{-(j-i-n)}A))\\&\leqq CM\sum_{\boldsymbol{k}_{i+n}\in\alpha}\lambda(B(\boldsymbol{k}_{i+n}))\lambda(T^{-(j-i-n)}A)\\&=CM\lambda(A\cap T^{-i}A)\lambda(T^{-(j-i-n)}A).\end{aligned}$$

Now let $B(\boldsymbol{a}_n)$, $B(\boldsymbol{b}_n)\subseteq A$. Then

$$\begin{aligned}\lambda(B(\boldsymbol{a}_n)\cap T^{-(j-i)}B(\boldsymbol{b}_n))&=\sum_{(k_{n+1},\cdots,k_{j-i})\in I^{j-i-n}}\lambda(B(\boldsymbol{a}_n,k_{n+1},\cdots,k_{j-i},\boldsymbol{b}_n))\\&\geqq M^{-1}\lambda(B(\boldsymbol{a}_n))\lambda(T^{-(j-i-n)}B(\boldsymbol{b}_n)).\end{aligned}$$

Summation over $\boldsymbol{a}_n$ and $\boldsymbol{b}_n$ gives

$$\lambda(A\cap T^{-(j-i)}A)\geqq M^{-1}\lambda(A)\lambda(T^{-(j-i-n)}A),$$

hence

$$\lambda(A\cap T^{-i}A\cap T^{-j}A)\leqq K_1\lambda(A\cap T^{-i}A)\lambda(A\cap T^{-(j-i)}A)$$

for $j>i+n$, where $K_1=CM^2\lambda(A)^{-1}$. For $i\leqq j\leqq i+n$,

$$\begin{aligned}\lambda(A\cap T^{-i}A\cap T^{-j}A)&\leqq\lambda(A\cap T^{-i}A)\\&\leqq K_2\lambda(A\cap T^{-i}A)\lambda(A\cap T^{-(j-i)}A),\end{aligned}$$

where $K_2^{-1}=\min\{\lambda(A\cap T^{-i}A):0\leqq i\leqq n\}$.

Putting $K=\max\{K_1,K_2\}$ we conclude

$$\begin{aligned}\int_A\left(\sum_{k=0}^{n-1}1_A\circ T^k\right)^2d\mu&\leqq 2d_1^{-2}d_2K\sum_{i=0}^{n-1}\sum_{j=i}^{n-1}\mu(A\cap T^{-i}A)\mu(A\cap T^{-(j-i)}A)\\&\leqq 2d_1^{-2}d_2Ka_n(A)^2.\end{aligned}$$

□

J. Aaronson has shown in [1] that the order of magnitude of the sequences $\{a_n(A)\}$ is the same for all $A\in B(T)$, and therefore an isomorphism invariant for rationally ergodic transformations, called the asymptotic type of T. For our transformations it seems easier to consider the minimal wandering rates, which will be studied in the next section. We refer to [3], §5, for general results concerning the connection of the asymptotic type and the minimal wandering rates.

3. Wandering rates

Let $T \in \mathcal{T} \setminus \mathcal{T}_R$ be fixed and let $L_A(n) = \mu(\bigcup_{k=0}^{n-1} T^{-k}A)$, $A \in \mathcal{R}$, be the wandering rate of the set A (cf. [3]). Furthermore, let

$$E(T) = \bigcup_{\varepsilon > 0} \left\{ A \in \mathcal{R} : \lambda(A) > 0, A \subseteq [0,1] \setminus \bigcup_{j \in J} (x_j - \varepsilon, x_j + \varepsilon) \right\}.$$

THEOREM 3. *$L_A(n) \sim L_B(n)$ $(n \to \infty)$ for all $A, B \in E(T)$.*

PROOF. Let $A = [0,1] \setminus \bigcup_{j \in J} B(j,j)$. Then

$$\bigcup_{k=0}^{n-1} T^{-k}A = [0,1] \setminus \bigcup_{j \in J} B(\underbrace{j, \cdots, j}_{n+1}),$$

hence by (1.6),

$$\begin{aligned} L_A(n) &= \mu\Big(\bigcup_{i \notin J} B(i)\Big) + \sum_{j \in J} \left\{ \int_{f_j(0)}^{f_j^{n+1}(0)} \frac{d\mu}{d\lambda}(x)\, d\lambda(x) + \int_{f_j^{n+1}(1)}^{f_j(1)} \frac{d\mu}{d\lambda}(x)\, d\lambda(x) \right\} \\ &\geqq c_1 \sum_{j \in J} \left\{ \int_{f_j(0)}^{f_j^{n+1}(0)} \frac{x - x_j}{x - f_j(x)}\, dx + \int_{f_j^{n+1}(1)}^{f_j(1)} \frac{x - x_j}{x - f_j(x)}\, dx \right\}. \end{aligned} \tag{3.1}$$

(Here f_j denotes the continuous extension of $(T_{|B(j)})^{-1}$ to $[0,1]$.)

Let first $B \subseteq A$, $\lambda(B) > 0$. Taking into account that

$$\bigcup_{k=0}^{n-1} T^{-k}A = A \cup \bigcup_{j \in J} \bigcup_{i \neq j} \bigcup_{k=2}^{n} B(\underbrace{j, \cdots, j}_{k}, i)$$

we get

$$\begin{aligned} \Big(\bigcup_{k=0}^{n-1} T^{-k}A\Big) \setminus \bigcup_{k=0}^{n-1} T^{-k}B &\subseteq A \setminus B \cup \bigcup_{j \in J} \bigcup_{k=1}^{n-1} \bigcup_{i \neq j} (B(\underbrace{j, \cdots, j}_{k}) \\ &\qquad \cap T^{-k}(B(j,i) \cap B^c)) \\ &\subseteq A \setminus B \cup \bigcup_{j \in J} \bigcup_{k=1}^{n-1} B(\underbrace{j, \cdots, j}_{k}) \cap T^{-k}(A \setminus B). \end{aligned}$$

Hence, applying (1.6) again,

$$L_A(n) \leqq L_B(n) + \mu(A \setminus B) + \sum_{j \in J} \sum_{k=1}^{n-1} \int_{A \setminus B} \frac{d\mu}{d\lambda}(f_j^k(x))(f_j^k)'(x)\,dx$$

$$\leqq L_B(n) + c_2 \int_{A \setminus B} \sum_{j \in J} \sum_{k=0}^{n-1} g_j(f_j^k(x))(f_j^k)'(x)\,dx, \tag{3.2}$$

where $g_j(x) = (x - x_j)/(x - f_j(x)), x \in [0,1] \setminus \{x_j\}$.

Let $j \in J$ be fixed and $x_j < 1$. By condition 3 of section 1 there exists a number η, $0 < \eta \leqq f_j(1) - x_j$, such that f_j' is decreasing on $(x_j, x_j + \eta)$. The derivative of g_j is given by

$$g_j'(x) = \left(\int_{x_j}^{x} (f_j'(x) - f_j'(t))\,dt \right) / (x - f_j(x))^2,$$

which shows that $g_j'(x) < 0$ for $x \in (x_j, x_j + \eta)$. Hence the function $\sum_{k=0}^{n-1} g_j(f_j^k(x))(f_j^k)'(x)$ is decreasing on $(x_j, x_j + \eta)$, and we obtain

$$(x - f_j(x)) \sum_{k=0}^{n-1} g_j(f_j^k(x))(f_j^k)'(x) \leqq \int_{f_j(x)}^{x} \sum_{k=0}^{n-1} g_j(f_j^k(t))(f_j^k)'(t)\,dt$$

$$= \sum_{k=0}^{n-1} \int_{f_j^{k+1}(x)}^{f_j^k(x)} g_j(t)\,dt$$

$$\leqq \int_{f_j^n(x)}^{f_j(1)} g_j(t)\,dt$$

for $x \in (x_j, x_j + \eta)$ and $n \geqq 1$.

Now choose $N = N(j) \geqq 1$ such that $f_j^N(x) \in (x_j, x_j + \eta)$ for all $x \in (x_j, 1)$, and $d(j) > 0$ such that $f_j^N(x) - f_j^{N+1}(x) \geqq d(j)^{-1}$ for all $x \in A \cap (x_j, 1)$. Since $\sum_{k=0}^{N-1} g_j(f_j^k(x))(f_j^k)'(x) \leqq c(j)$ on A for some constant $c(j)$ we get for $x \in A \cap (x_j, 1)$ and $n > N$,

$$\sum_{k=0}^{n-1} g_j(f_j^k(x))(f_j^k)'(x) \leqq c(j) + \sum_{k=N}^{n-1} g_j(f_j^k(x))(f_j^k)'(x)$$

$$\leqq c(j) + \sum_{k=0}^{n-N-1} g_j(f_j^k(f_j^N(x)))(f_j^k)'(f_j^N(x))$$

$$\leqq c(j) + d(j) \int_{f_j^n(x)}^{f_j(1)} g_j(t)\,dt$$

$$\leqq c(j) + d(j) \int_{f_j^{n+2}(1)}^{f_j(1)} g_j(t)\,dt,$$

since $x \in A \cap (x_j, 1)$ implies $f_j^2(1) \leqq x$ and hence $f_j^{n+2}(1) \leqq f_j^n(x)$. In view of (3.1) we get

$$\sum_{k=0}^{n-1} g_j(f_j^k(x))(f_j^k)'(x) \leqq c(j) + d(j)c_1^{-1} L_A(n+1) \tag{3.3}$$

for all $x \in A \cap (x_j, 1)$ and all $n \geqq 1$.

Applying the same argument to $x \in A \cap (0, x_j)$, if $x_j > 0$, we see that (3.3) holds for all $x \in A$ (with possibly larger constants). Since J is finite,

$$\sum_{j \in J} \sum_{k=0}^{n-1} g_j(f_j^k(x))(f_j^k)'(x) \leqq K(1 + L_A(n+1)), \qquad x \in A,$$

for some constant K, and (3.2) implies

$$L_A(n) \leqq L_B(n) + Kc_2(1 + L_A(n+1))\lambda(A \setminus B).$$

This shows that

$$\underline{\lim}\, L_B(n)/L_A(n) \geqq 1 - c_2 K\lambda(A \setminus B). \tag{3.4}$$

Now let $B_s = \bigcup_{k=0}^{s} T^{-k}B$. Then $L_{B_s}(n) \leqq L_B(n) + s\mu(B)$, and we get by applying (3.4) to $A \cap B_s$,

$$\begin{aligned} \underline{\lim}\, L_B(n)/L_A(n) &\geqq \underline{\lim}\,(L_B(n)/(L_B(n) + s\mu(B)))(L_{A \cap B_s}(n)/L_A(n)) \\ &\geqq 1 - c_2 K\lambda(A \setminus B_s) \qquad \text{for all } s \geqq 1. \end{aligned}$$

Taking into account that $\lambda(A \setminus B_s) \to 0$ as $s \to \infty$ we see that

$$\lim_{n \to \infty} L_B(n)/L_A(n) = 1 \qquad \text{for all } B \subseteq A \quad \text{with } \lambda(B) > 0.$$

Now let $B \in E(T)$ be arbitrary. Then

$$A \cap T^{-2}B \subseteq T^{-2}B \subseteq \bigcup_{k=0}^{N} T^{-k}A \qquad \text{for some } N \geqq 1.$$

Since $\lambda(A \cap T^{-2}B) > 0$ and $L_B(n) = L_{T^{-2}B}(n) \leqq L_A(n) + N\mu(A)$ we conclude

$$\begin{aligned} 1 &= \lim_{n \to \infty} L_{A \cap T^{-2}B}(n)/L_A(n) \leqq \underline{\lim}\, L_B(n)/L_A(n) \\ &\leqq \overline{\lim}\, L_B(n)/L_A(n) \leqq \lim_{n \to \infty} (L_A(n) + N\mu(A))/L_A(n) = 1. \qquad \square \end{aligned}$$

From a probabilistic point of view the statement of Theorem 3 may be interpreted as follows. Let $0 < \mu(A) < \infty$ and $n(x) = \min\{n \geqq 1 : T^n(x) \in A\}$. Define the stopping time τ_n by

$$\tau_n(x) = \begin{cases} n(x), & \text{if } n(x) \leqq n, \\ n, & \text{if } n(x) > n, \end{cases}$$

i.e. the process $(Tx, T^2x, \cdots)$ stops the first time it reaches the set A but after n steps at the latest. Then the expected number $m_A(n)$ of steps when starting in A is given by

$$\begin{aligned} m_A(n) &= (1/\mu(A))\int_A \tau_n(x)\,d\mu(x) \\ &= (1/\mu(A))\sum_{k=1}^{n} \mu(\{x \in A : n(x) \geqq k\}) \\ &= (1/\mu(A))(\mu(A) + \sum_{k=1}^{n-1} \mu(\{x \in A^c : n(x) = k\})) \\ &= L_A(n)/\mu(A). \end{aligned}$$

Hence the theorem asserts that $m_A(n)/m_B(n) \to \mu(B)/\mu(A)$ as $n \to \infty$ for all $A, B \in E(t)$.

We shall call the rate of growth of the sequences $\{L_A(n)\}$, $A \in E(T)$, the wandering rate of T. By $\{w_n(T)\}$ we denote any sequence for which $w_n(T) \sim L_A(n)$ $(n \to \infty)$ for one — and hence for all — $A \in E(T)$.

PROPOSITION. *Let T_1, $T_2 \in \mathcal{T} \setminus \mathcal{T}_R$ be weakly isomorphic via $\phi : T_1 \to T_2$ and $\psi : T_2 \to T_1$, where $\mu_1 \circ \phi^{-1} = c\mu_2$ $(c > 0)$. Then,*

$$w_n(T_1) \sim c w_n(T_2) \qquad \text{as } n \to \infty.$$

PROOF. Choose $A \in E(T_2)$ and $B \in E(T_1)$ with $B \subseteq \phi^{-1}(A)$. Then,

$$\begin{aligned} L_B(n) \leqq L_{\phi^{-1}(A)}(n) &= \mu_1\left(\bigcup_{k=0}^{n-1} T_1^{-k}\phi^{-1}(A)\right) \\ &= \mu_2\left(\phi^{-1}\left(\bigcup_{k=0}^{n-1} T_2^{-k}A\right)\right) = cL_A(n), \end{aligned}$$

hence $\overline{\lim}\, w_n(T_1)/w_n(T_2) = \overline{\lim}\, L_B(n)/L_A(n) \leqq c$.

On the other hand, $\mu_2 \circ \psi^{-1} = c'\mu_1$ for some constant c'. Since T_1, T_2 are rationally ergodic we know from [1] that $c' = c^{-1}$. Therefore the same reasoning as above yields

$$\underline{\lim}\, w_n(T_1)/w_n(T_2) = 1/(\overline{\lim}\, w_n(T_2)/w_n(T_1)) \geqq c. \qquad \square$$

Our next goal is to calculate the wandering rate of transformations $T \in \mathcal{T} \setminus \mathcal{T}_R$ admitting expansions at the critical fixed points. We shall proceed by stating five lemmas, the second, third and fourth of which are of some independent interest. The proof of Lemma 1 is a standard ε-δ-argument, and is therefore omitted.

LEMMA 1. *Let $f, g : (a, b] \to \mathbf{R}$ be nonnegative and integrable on $[a+\delta, b]$ for each $\delta > 0$, $\int_a^b f d\lambda = \infty$ and $f(x) \sim g(x)$ as $x \to a$. Then,*

$$\int_y^b f d\lambda \sim \int_y^b g d\lambda \qquad \text{as } y \to a.$$

LEMMA 2. *Let $f : [0, \eta] \to \mathbf{R}$ ($\eta > 0$) be differentiable and concave satisfying $0 < f(x) < x$, $0 < x \leqq \eta$, and $f'(0) = 1$. Let $g : [0, \eta] \to \mathbf{R}$ be nonnegative and bounded such that the function $g(x)/(x - f(x))$ is decreasing on $(0, \eta]$; let $a_0 \in (0, \eta]$, $a_k = f^k(a_0)$ ($k \geqq 1$). Then,*

(a) *if $\int_0^\eta g(x)/(x - f(x))dx < \infty$, $\sum_{k=0}^\infty g(a_k)$ converges;*

(b) *if $\int_0^\eta g(x)/(x - f(x))dx = \infty$, $\int_{a_n}^\eta g(x)/(x - f(x))dx \sim \sum_{k=0}^{n-1} g(a_k)$ as $n \to \infty$.*

PROOF. By the conditions imposed on f the sequence $\{a_k\}$ is decreasing and $\lim_{k\to\infty} a_k = 0$. Taking into account that

$$a_{k+1} - f(a_{k+1}) = f(a_k) - f(a_{k+1}) = f'(\xi_k)(a_k - a_{k+1})$$

for some $\xi_k \in (a_{k+1}, a_k)$ the concavity of f implies

$$(a_k - a_{k+1})/(a_{k+1} - f(a_{k+1})) \leqq 1/f'(a_k), \qquad k \geqq 0.$$

Therefore,

$$\begin{aligned} g(a_k) &= \{g(a_k)/(a_k - f(a_k))\}(a_k - a_{k+1}) \\ &\leqq \int_{a_{k+1}}^{a_k} g(x)/(x - f(x))dx \\ &\leqq \{g(a_{k+1})/(a_{k+1} - f(a_{k+1}))\}(a_k - a_{k+1}) \\ &\leqq g(a_{k+1})/f'(a_k) \end{aligned}$$

for $k \geqq 0$. Summation over $0, 1, \cdots, n-1$ then yields

$$\begin{aligned} \sum_{k=0}^{n-1} g(a_k) &\leqq \int_{a_n}^{a_0} g(x)/(x - f(x))dx \\ &\leqq \sum_{k=0}^{n-1} g(a_{k+1})/f'(a_k) \qquad (n \geqq 1). \end{aligned}$$

From this the first assertion follows immediately.

If $\int_0^\eta g(x)/(x - f(x))dx = \infty$, the upper inequality shows that $\sum_{k=0}^\infty g(a_k)$ diverges. Since $\lim_{k\to\infty} f'(a_k) = 1$ and g is bounded

$$\sum_{k=0}^{n-1} g(a_{k+1})/f'(a_k) \sim \sum_{k=0}^{n-1} g(a_k) \qquad (n \to \infty)$$

holds true. This completes the proof. □

If f satisfies the conditions of Lemma 2, we may always take $g(x)\equiv x$ and $g(x)\equiv 1$. The first yields

$$\int_{a_n}^{\eta} x/(x-f(x))\,dx \sim \sum_{k=0}^{n-1} a_k \quad (n\to\infty), \qquad \text{if} \quad \int_0^{\eta} x/(x-f(x))dx=\infty.$$

The second gives the interesting relation

$$\int_{a_n}^{\eta} \frac{dx}{x-f(x)} \sim n \qquad \text{as } n\to\infty,$$

which determines implicitly the order of magnitude of the sequence $\{a_n\}$.

COROLLARY. *Let f satisfy the conditions of Lemma 2 and let $f(x)= x-ax^{p+1}+o(x^{p+1})$ as $x\to 0$ $(a>0, p>0)$. Then,*

$$a_n \sim 1/(apn)^{1/p} \qquad \text{as } n\to\infty.$$

PROOF. Using Lemma 1 we get

$$n\sim\int_{a_n}^{\eta}\frac{dx}{x-f(x)}\sim\int_{a_n}^{\eta}\frac{dx}{ax^{p+1}}=\frac{a_n^{-p}-\eta^{-p}}{ap}\sim\frac{a_n^{-p}}{ap}. \qquad □$$

LEMMA 3. *For $T\in\mathcal{T}_R$ there exists a version of $d\mu/d\lambda$ which is continuous on* $[0,1]$.

PROOF. The proof is a modification of the argument used in [8], theorem 8, where the assertion is proved for finite-to-one transformations (see also section 7 in [8]).

Let $I=\{1,2,\cdots\}$ be finite or infinite. From $T(f_k(x))=x$, $x\in(0,1)$, and condition (4) imposed on T we obtain

$$\frac{|f_k''(x)|}{f_k'(x)}=\frac{|T''(f_k(x))|}{T'(f_k(x))^2}\leqq M, \qquad x\in(0,1), \quad k\geqq 1.$$

For $(k_1,\cdots,k_n)\in I^n$ and $x\in(0,1)$,

$$\frac{|f''_{k_1,\cdots,k_n}(x)|}{f'_{k_1,\cdots,k_n}(x)}\leqq\sum_{i=1}^{n-1}\frac{|f''_{k_i}(f_{k_{i+1},\cdots,k_n}(x))|}{f'_{k_i}(f_{k_{i+1},\cdots,k_n}(x))}f'_{k_{i+1},\cdots,k_n}(x)+\frac{|f''_{k_n}(x)|}{f'_{k_n}(x)}$$

$$\leqq M\cdot(1+\rho+\cdots+\rho^{n-1})\leqq M/(1-\rho)=K \qquad (\rho<1).$$

In particular, $f''_{k_1,\cdots,k_n}$ is bounded on $(0,1)$. Together with the other conditions for T this implies that the functions $f_{k_1,\cdots,k_n}$ have C^1-extensions to $[0,1]$.

By the mean value theorem,

$$\left|\log \frac{f'_{k_1,\cdots,k_n}(x)}{f'_{k_1,\cdots,k_n}(y)}\right| \leqq K \qquad \text{for } x, y \in [0,1].$$

Hence there exists a constant C such that

(i) $(1/C)\lambda(B(k_1,\cdots,k_n)) \leqq f'_{k_1,\cdots,k_n}(x) \leqq C\lambda(B(k_1,\cdots,k_n))$, $x \in [0,1]$, and

(ii) $|f''_{k_1,\cdots,k_n}(x)| \leqq C\lambda(B(k_1,\cdots,k_n))$, $x \in (0,1)$, for all $(k_1,\cdots,k_n) \in I^n$ and all $n \geqq 1$.

Now let the functions h_n, $n \geqq 0$, be defined on [0,1] by

$$h_0 = 1, \qquad h_{n+1} = \sum_{k \geqq 1} (h_n \circ f_k) \cdot f'_k \qquad (n \geqq 0),$$

i.e. $h_{n+1} = Ah_n$, where A is the Frobenius–Perron operator on $L_1([0,1],\lambda)$ associated with T. Then,

$$h_n(x) = \sum_{k_1 \geqq 1} \cdots \sum_{k_n \geqq 1} f'_{k_1,\cdots,k_n}(x), \qquad x \in [0,1].$$

From (i) it follows that these series are uniformly convergent on [0, 1]. Hence the functions h_n are continuous on [0,1], and $1/C \leqq h_n \leqq C$. By (ii),

$$|f'_{k_1,\cdots,k_n}(x) - f'_{k_1,\cdots,k_n}(y)| \leqq C \cdot |x - y| \cdot \lambda(B(k_1,\cdots,k_n))$$

for all $(k_1,\cdots,k_n) \in I^n$, hence

$$|h_n(x) - h_n(y)| \leqq C \cdot |x - y| \qquad \text{for all } x, y \in [0,1]$$

and all $n \geqq 0$. This implies that the sequence

$$\left\{(1/n)\sum_{j=0}^{n-1} h_j\right\}_{n=1}^{\infty}$$

is equicontinuous on [0,1]. By the theorem of Arzelà–Ascoli there exist a subsequence (n_i) and a continuous function h such that

$$g_{n_i} := (1/n_i)\sum_{j=0}^{n_i-1} h_j \to h \qquad \text{uniformly on } [0,1] \text{ as } i \to \infty.$$

Since $Ag_{n_i} = g_{n_i} + (1/n_i)(h_{n_i} - 1)$ and $h_{n_i} \leqq C$, $Ag_{n_i} \to h$ as $i \to \infty$. On the other hand, from

$$|Ag_{n_i}(x) - Ah(x)| \leqq \sum_{k \geqq 1} |g_{n_i}(f_k(x)) - h(f_k(x))| \cdot f'_k(x)$$

$$\leqq C \cdot \max_{x \in [0,1]} |g_{n_i}(x) - h(x)|$$

we see that $Ag_{n_i} \to Ah$ as $i \to \infty$. Therefore $Ah = h$, i.e. h is a continuous version of $d\mu/d\lambda$ (with $1/C \leqq h \leqq C$). □

LEMMA 4. *Let $T \in \mathcal{T} \setminus \mathcal{T}_R$. Then there exists a continuous function $g = g_\mu$ on $[0,1]$ such that*

$$\frac{d\mu}{d\lambda}(x) = g(x) \prod_{j \in J} \frac{x - x_j}{x - f_j(x)}, \qquad x \in [0,1] \setminus \{x_j : j \in J\}.$$

PROOF. By (1.3) the invariant density can be written in the form

$$\frac{d\mu}{d\lambda}(x) = h^*(x) + \sum_{j \in J} \sum_{k=1}^{\infty} h^*(f_j^k(x))(f_j^k)'(x),$$

where h^* is a version of the invariant density of T^*. According to Lemma 3 we may suppose h^* to be continuous on $[0,1]$. Define g on $[0,1] \setminus \{x_j : j \in J\}$ by

$$g(x) = \frac{d\mu}{d\lambda}(x) \prod_{j \in J} \frac{x - f_j(x)}{x - x_j}$$

with this version of h^*.

Let $c > 0$ be a constant such that $h^*(x) \leqq c$ for $x \in [0,1]$, and let $j \in J$ be fixed, $x_j < 1$ and $0 < \varepsilon < 1 - x_j$. Choose $0 < \eta < 1 - x_j$, $N \geqq 0$ and $d > 0$ such that

$$f_j' \text{ is decreasing on } (x_j, x_j + \eta),$$

$$f_j^N(x) \in (x_j, x_j + \eta) \qquad \text{for all } x \in (x_j, 1],$$

and

$$f_j^N(x) - f_j^{N+1}(x) \geqq 1/d \qquad \text{for all } x \in (x_j + \varepsilon, 1].$$

As in the proof of Theorem 3 integration over the interval $[f_j(x), x]$ yields

$$(x - f_j(x)) \sum_{k=0}^{\infty} (f_j^{s+k})'(x) \leqq f_j^s(x) - x_j$$

for all $x \in (x_j, x_j + \eta)$ and all $s \geqq 0$. Therefore we obtain for $n \geqq N$ and $x \in (x_j + \varepsilon, 1]$

$$\sum_{k=n}^{\infty} h^*(f_j^k(x))(f_j^k)'(x) \leqq c \sum_{k=0}^{\infty} (f_j^{n+k})'(x)$$

$$\leqq c \sum_{k=0}^{\infty} (f_j^{n-N+k})'(f_j^N(x)) \leqq c \cdot d\,(f_j^n(x) - x_j) \leqq c \cdot d\,(f_j^n(1) - f_j^n(0)).$$

Applying the same argument to the left hand side of x_j, if $x_j > 0$, we see that the series $\sum_{k=1}^{\infty} h^*(f_j^k(x))(f_j^k)'(x)$ is uniformly convergent on $[0,1] \setminus (x_j - \varepsilon, x_j + \varepsilon)$ for every $\varepsilon > 0$ and every $j \in J$. Hence g is continuous on $[0,1] \setminus \{x_j : j \in J\}$.

It remains to show that $\lim_{x\to x_j} g(x)$ exists for each $j \in J$. Let $j \in J$ be fixed, and choose $\varepsilon > 0$, $\delta > 0$ such that

$$|h^*(x) - h^*(x_j)| \leqq \varepsilon \qquad \text{for } |x - x_j| \leqq \delta.$$

Since $|f_j^k(x) - x_j| \leqq |x - x_j|$ this implies

$$|h^*(f_j^k(x)) - h^*(x_j)| \leqq \varepsilon \qquad \text{for } |x - x_j| \leqq \delta \quad \text{and all } k \geqq 1.$$

Therefore,

$$\left| \sum_{k=1}^{\infty} h^*(f_j^k(x))(f_j^k)'(x) - h^*(x_j) \sum_{k=1}^{\infty} (f_j^k)'(x) \right| \leqq \varepsilon \sum_{k=1}^{\infty} (f_j^k)'(x)$$

for $|x - x_j| \leqq \delta$, and hence

$$\lim_{x\to x_j} \sum_{k=1}^{\infty} h^*(f_j^k(x))(f_j^k)'(x) \Big/ \left(\sum_{k=1}^{\infty} (f_j^k)'(x) \right) = h^*(x_j).$$

Taking into account that the functions $\sum_{k=1}^{\infty} h^*(f_i^k(x))(f_i^k)'(x)$ are bounded on $B(j)$ for all $i \in J \setminus \{j\}$ and

$$\lim_{x\to x_j} \sum_{k=1}^{\infty} (f_j^k)'(x) = \infty$$

(cf. [20]) we obtain

$$\lim_{x\to x_j} \frac{d\mu}{d\lambda}(x) \Big/ \left(\sum_{k=1}^{\infty} (f_j^k)'(x) \right) = h^*(x_j).$$

Finally, by the lemma in [20],

$$\lim_{x\to x_j} \frac{x - f_j(x)}{x - x_j} \sum_{k=1}^{\infty} (f_j^k)'(x) = 1.$$

Hence

$$\lim_{x\to x_j} g(x) =: g(x_j)$$

exists. □

LEMMA 5. *Let g, h^* be as in Lemma 4. Then,*

$$w_n(T) \sim \sum_{j\in J} h^*(x_j) \sum_{k=1}^{n} (f_j^k(1) - f_j^k(0)) \qquad \text{as } n \to \infty,$$

and

$$h^*(x_j) = g(x_j) \prod_{\substack{i\in J\\ i\neq j}} \frac{x_j - x_i}{x_j - f_i(x_j)}, \qquad i \in J.$$

PROOF. Let $A=[0,1]\setminus\bigcup_{j\in J}B(j,j)$. By Lemma 4,

$$\frac{d\mu}{d\lambda}(x)\sim h^*(x_j)\frac{x-x_j}{x-f_j(x)}\qquad \text{as } x\to x_j.$$

Hence applying Lemma 1 we get

$$\begin{aligned} w_n(T)&\sim L_A(n)\\ &=\mu\Big(\bigcup_{k=0}^{n-1}T^{-k}A\Big)\\ &=\mu\Big(\bigcup_{i\notin J}B(i)\Big)+\sum_{j\in J}\big(\mu([f_j(0),f_j^{n+1}(0)])+\mu([f_j^{n+1}(1),f_j(1)])\big)\\ &\sim\sum_{j\in J}h^*(x_j)\Big(\int_0^{f_j^n(0)}\frac{x-x_j}{x-f_j(x)}dx+\int_{f_j^n(1)}^1\frac{x-x_j}{x-f_j(x)}dx\Big). \end{aligned}$$

From this the result follows by applying Lemma 2 (with $g(x)\equiv x$) to $f(x)=x_j-f_j(x_j-x)$ resp. $f(x)=f_j(x_j+x)-x_j$. The formula connecting $h^*(x_j)$ and $g(x_j)$ results from the proof of Lemma 4. □

Now assume that for each $j\in J$

$$T(x)=x\pm a_j(x-x_j)^{p_j+1}+o((x-x_j)^{p_j+1})\qquad \text{as } x\to x_j,$$

or equivalently

$$f_j(x)=x\mp a_j(x-x_j)^{p_j+1}+o((x-x_j)^{p_j+1})\qquad \text{as } x\to x_j,$$

where $a_j>0$, $p_j\in\mathbf{N}$.

Then

$$\frac{d\mu}{d\lambda}(x)=\prod_{j\in J}|x-x_j|^{-p_j}h(x),$$

where by Lemma 4, $h(x)$ may be assumed to be continuous on $[0,1]$.

With the notations

$$p=\max\{p_j:j\in J\},$$

$$J_0=\{j\in J:p_j=p\},$$

$$\varepsilon(x)=\begin{cases}1, & \text{if } x\in\{0,1\},\\ 2, & \text{if } 0<x<1,\end{cases}$$

$$c_j=\prod_{\substack{i\in J\\ i\neq j}}|x_j-x_i|^{-p_i}h(x_j),$$

we obtain

THEOREM 4.

$$w_n(T) = \begin{cases} \left(\sum_{j \in J} h(x_j)\right) \log n, & \text{if } p = 1, \\ \dfrac{p^{1-1/p}}{p-1}\left(\sum_{j \in J_0} \varepsilon(x_j) c_j a_j^{1-1/p}\right) n^{1-1/p}, & \text{if } p > 1. \end{cases}$$

PROOF. By the corollary to Lemma 2,

$$f_j^k(1) - f_j^k(0) = (f_j^k(1) - x_j) + (x_j - f_j^k(0))$$
$$\sim \varepsilon(x_j)(a_j p_j k)^{-1/p_j}$$

as $k \to \infty$, hence

$$\sum_{k=1}^{n} (f_j^k(1) - f_j^k(0)) \sim \begin{cases} \varepsilon(x_j) a_j^{-1} \log n & \text{if } p_j = 1, \\ \varepsilon(x_j)(a_j p_j)^{-1/p_j} \dfrac{n^{1-1/p_j}}{1 - 1/p_j} & \text{if } p_j > 1. \end{cases}$$

Taking into account that

$$h^*(x_j) = h(x_j) a_j \prod_{\substack{i \in J \\ i \neq j}} |x_j - x_i|^{-p_i}$$

the assertion now follows from Lemma 5. Notice that the numbers p_j are even, if $0 < x_j < 1$. Hence $J_0 = J \subseteq \{0, 1\}$ when $p = 1$. □

Before giving some examples we indicate a possible extension of the definition of the wandering rate to more general transformations.

Let $(X, \mathcal{R}, \mu)$ be a σ-finite measure space, and let $T: X \to X$ be measure preserving, conservative and ergodic. Motivated by the proof of Theorem 3 we define the class $W(T)$ by

$$W(T) = \{A \in \mathcal{R} : 0 < \mu(A) < \infty, L_B(n) \sim L_A(n) \text{ as } n \to \infty$$
$$\text{for all } B \in A \cap \mathcal{R} \text{ with } \mu(B) > 0\}.$$

Then the following assertions hold:

(i) $A \in W(T)$ iff $0 < \mu(A) < \infty$ and $\underline{\lim}\, L_B(n)/L_A(n) \geqq 1$ for all $B \in \mathcal{R}$ with $\mu(B) > 0$,

(ii) $L_A(n) \sim L_B(n)$ as $n \to \infty$ for all $A, B \in W(T)$,

(iii) if T_1, T_2 are isomorphic by $\phi: T_1 \overset{c}{\to} T_2$, then $\phi(W(T_1)) = W(T_2)$ and $L_A(n) \sim c L_B(n)$ as $n \to \infty$ for all $A \in W(T_1)$, $B \in W(T_2)$.

Hence, for all T with $W(T) \neq \phi$ the wandering rate can be defined as the rate of growth of the sequences $\{L_A(n)\}$, $A \in W(T)$.

EXAMPLES.

(1)
$$T(x) = \begin{cases} x/(1-x), & x \in B(0) = [0, 1/2], \\ 2 - 1/x, & x \in B(1) = (1/2, 1], \end{cases}$$

$$\frac{d\mu}{d\lambda}(x) = 1/x(1-x).$$

Here we have $J = J_0 = \{0, 1\}$, $p = 1$, $h(0) = h(1) = 1$, hence

$$w_n(T) = 2 \log n,$$

as in this case, can easily be verified also by a direct calculation.

(2) $Tx = \tan x$, $x \in \mathbf{R}$, $d\mu(x)/d\lambda = 1/x^2$.
Consider $S = \phi T \phi^{-1}$ where $\phi(x) = (1/\pi) \arctan x + \frac{1}{2}$. Since

$$S(x) = x + \frac{\pi^2}{3}\left(x - \frac{1}{2}\right)^3 + \cdots$$

on $B(0) = (\frac{1}{2} - (1/\pi) \arctan \pi/2, \frac{1}{2} + (1/\pi) \arctan \pi/2)$, and the invariant density of S is given by

$$\frac{d\mu}{d\lambda}(\phi^{-1}(x))(\phi^{-1})'(x) = \pi/\cos^2 \pi x = h(x)/(x - \tfrac{1}{2})^2 \quad \text{with } h(\tfrac{1}{2}) = 1/\pi,$$

we have $p = 2$, $c_0 = 1/\pi$, $\varepsilon(\frac{1}{2}) = 2$, $a_0 = \pi^2/3$. Thus,

$$w_n(T) = w_n(S) = 2\sqrt{2n/3}.$$

(3) $Tx = x + \sum_{s=1}^{N} p_s/(\eta_s - x)$, $x \in \mathbf{R}$ $(p_s > 0)$, $d\mu/d\lambda = 1$.
By transforming on $[0,1]$ with $\phi(x) = (1/\pi) \arctan x + \frac{1}{2}$ one gets a transformation S with critical fixed points $x_0 = 0$ and $x_1 = 1$, and invariant density

$$(\phi^{-1})'(x) = \pi/\sin^2 \pi x = h(x)/x^2(x-1)^2, \qquad h(0) = h(1) = 1/\pi.$$

The expansions at x_0 and x_1 are

$$S(x) = x + a\pi^2 x^3 + \cdots \quad \text{resp.} \quad S(x) = x + a\pi^2(x-1)^3 + \cdots,$$

where $a = \sum_{s=1}^{N} p_s$. Hence, $w_n(T) = 2\sqrt{2an}$.

As proved in [2], $a_n(A)/\mu(A) \sim (1/\pi)\sqrt{2n/a}$ for $A \in B(T)$. Thus theorem 3 in [3] may be applied to confirm our result.

(4) We close with an S-unimodal function taken from [19] (cf. also [6]):

$$T(x)=\frac{1-5x^2}{1+3x^2},\quad x\in[-1,1],$$

$$\frac{d\mu}{d\lambda}(x)=(1+x)^{-1}(1-x^2)^{-1/2}.$$

Consider $S=\phi T\phi^{-1}$ on $[0,1]$, where $\phi(x)=(1/\pi)\arccos(-x)$. The expansion of S at the critical point 0 is given by

$$S(x)=x+(\pi^2/4)x^3+\cdots,$$

and

$$\frac{d\mu\circ\phi^{-1}}{d\lambda}(x)=\pi/(1-\cos\pi x)=(1/x^2)h(x)\qquad\text{with } h(0)=2/\pi.$$

Thus, $w_n(T)=\sqrt{2n}$.

We should note that in Examples (3) and (4) $|S'(x)|$ is equal to 1 for points x different from the fixed points x_j, $j\in J$. This difficulty, however, is easily removed by considering $(S^*)^2$ instead of S^*.

4. Entropy and McMillan's Theorem

Let us first recall the definition of the entropy of conservative transformations as given in [11]. Let $(X,\mathcal{R},\mu)$ be a σ-finite measure space and $T:X\to X$ be conservative, ergodic and measure preserving. If A_1, A_2 are sets of positive finite measure, it follows from $T_{A_i}=(T_{A_1\cup A_2})_{A_i}$ and the entropy formula for induced transformations that

$$\mu(A_i)h(T_{A_i})=\mu(A_1\cup A_2)h(T_{A_1\cup A_2})\qquad(i=1,2),$$

where h denotes the entropy with respect to the corresponding normalized measure. Thus the number

$$\boldsymbol{h}(T,\mu)=\mu(A)h(T_A),\qquad 0<\mu(A)<\infty,$$

is independent of A. It is defined as the entropy of T with respect to μ. Note that $\boldsymbol{h}(T,\mu)=\mu(X)h(T)$ if $\mu(X)<\infty$.

For $T\in\mathcal{T}_R$, $\boldsymbol{h}(T,\mu)$ is given by Rohlin's well-known formula

$$\boldsymbol{h}(T,\mu)=\int_0^1\log T'(x)\,d\mu(x)\qquad\text{(cf. [15]).}\tag{4.1}$$

By McMillan's Theorem, if $\boldsymbol{h}(T,\mu)<\infty$, (4.2) $$-\lim_{n\to\infty}(1/n)\log\lambda(B(\mathbf{k}_n(x))) = \mathbf{h}(T,\mu)/\mu([0,1]) \qquad \text{a.e.}$$

where $B(\mathbf{k}_n(x))$ is the cylinder of order n containing x.

We shall show that (4.1) is true for all $T\in\mathcal{T}$, and also (4.2) when suitably modified. Furthermore, (1.6) will yield a simple criterion for the finiteness of $\mathbf{h}(T,\mu)$.

Let $T\in\mathcal{T}$, and $T_{A,n}$ be an auxiliary transformation as defined in section 1 such that the sets D_n are unions of cylinders of order $n+s$ for a fixed $s\in\{-1,0,1,2,\cdots\}$, i.e.

$$D_n = \bigcup_{\mathbf{k}_{n+s}\in\Delta_n} B(\mathbf{k}_{n+s}) \qquad (\mathbf{k}_{n+s}=(k_1,\cdots,k_{n+s})),\quad n\geqq 1,$$

for a suitable index set Δ_n ($D_1=[0,1]$ for $s=-1$).

Let ν be invariant for $T_{A,n}$, $\nu\ll\lambda$, and let μ be given by (1.3). Then

$$\frac{d\mu}{d\lambda}(x)=\sum_{n=1}^{\infty}\sum_{\mathbf{k}_{n+s}\in\Delta_n}\omega_\nu(\mathbf{k}_{n+s})(x),$$

where

$$\omega_\nu(\mathbf{k}_t)(x)=\frac{d\nu}{d\lambda}(f_{k_t}(x))\omega(\mathbf{k}_t)(x),\qquad t\geqq 1,$$

$$\omega_\nu(\mathbf{k}_0)(x)=\frac{d\nu}{d\lambda}(x).$$

Under these assumptions we have the following

LEMMA. $\int_0^1\log T'(x)\,d\mu(x)=\int_A\log T'_{A,n}(x)\,d\nu(x)$.

PROOF. Assume one of these integrals is finite. Then

$$\int_A \log(T^{s+1})'(x)\,d\nu(x) \quad \text{is finite.}$$

To prove this, let first the right hand integral be finite. From

$$(T^{s+1})'(x)\leqq(T^{s+1}_{A,n})'(x) \qquad \text{a.e. on } A$$

and the fact that ν is invariant for $T_{A,n}$ it follows that

$$\begin{aligned}\int_A\log(T^{s+1})'(x)\,d\nu(x) &\leqq \int_A \log(T^{s+1}_{A,n})'(x)\,d\nu(x)\\ &=\sum_{i=0}^{s}\int_A\log T'_{A,n}(T^i_{A,n}x)\,d\nu(x)\\ &=(s+1)\int_A\log T'_{A,n}(x)\,d\nu(x)<\infty.\end{aligned}$$

If the left hand integral is finite,

$$\infty > (s+1)\int_0^1 \log T'(x)\,d\mu(x) = \int_0^1 \log(T^{s+1})'(x)\,d\mu(x)$$

$$\geqq \int_A \log(T^{s+1})'(x)\,d\mu(x) \geqq \int_A \log(T^{s+1})'(x)\,d\nu(x),$$

since $\mu(E) \geqq \nu(E)$ for all $E \in A \cap \mathcal{R}$.

This consideration justifies the following calculation:

$$\int_0^1 \log T'(x)\,d\mu(x) = \sum_{n=1}^{\infty} \sum_{k_{n+s}\in\Delta_n} \int_0^1 \log T'(x)\omega_\nu(k_{n+s})(x)\,d\lambda(x)$$

$$= \sum_{n=1}^{\infty} \int_{D_n} \log T'(T^{n+s}(x))\,d\nu(x)$$

$$= \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \int_{B_k} \log T'(T^{n+s}(x))\,d\nu(x)$$

$$= \sum_{k=1}^{\infty} \int_{B_k} \log(T^k)'(x)\,d\nu(x)$$

$$+ \sum_{k=1}^{\infty} \int_{B_k} (\log(T^{s+1})'(T^k(x)) - \log(T^{s+1})'(x))\,d\nu(x)$$

$$= \int_A \log T'_{A,n}(x)\,d\nu(x)$$

$$+ \int_A \log(T^{s+1})'(T_{A,n}(x))\,d\nu(x) - \int_A \log(T^{s+1})'(x)\,d\nu(x)$$

$$= \int_A \log T'_{A,n}(x)\,d\nu(x). \qquad \square$$

Now we can prove

THEOREM 5. *Let $T \in \mathcal{T}$.*

(1) *$h(T,\mu) = \int_0^1 \log T'(x)\,d\mu(x) = \nu([0,1])h(T^*)$, if ν is the invariant measure of T^* and $\mu(E) = \sum_{k=1}^{\infty} \nu(T^{-k}E \cap D_k)$, $E \in \mathcal{R}$.*

(2) *Let $u_j(x) = (x - x_j)(T'(x) - 1)(Tx - x)^{-1}$, $x \in B(j)$, $j \in J$. If*

$$-\sum_{i\in I} \lambda(B(i)) \log \lambda(B(i)) < \infty,$$

then $h(T,\mu) < \infty$, if and only if $u_j \in L_1(B(j),\lambda)$ for all $j \in J$.

(3) *If* $\boldsymbol{h}(T,\mu)<\infty$, *then*

$$-\lim_{n\to\infty}\log\lambda(B(\boldsymbol{k}_n(x)))\Big/\sum_{k=0}^{n-1}g(T^kx)=\boldsymbol{h}(T,\mu)\Big/\int_0^1 g\,d\mu \qquad a.e.$$

for all $g\in L_1(\mu)$, $g\geqq 0$, $\int_0^1 g\,d\mu>0$.

PROOF. (1) Let $A=B(k)$, $k\in I\setminus J$. Then $0<\mu(A)<\infty$, $T_A\in\mathcal{T}_R(A)$ and $\mu_{|A\cap\mathcal{B}}$ is invariant for T_A. Hence by (4.1) and the Lemma,

$$\boldsymbol{h}(T,\mu)=\mu(A)\boldsymbol{h}(T_A)=\int_A \log T_A'(x)\,d\mu(x)=\int_0^1\log T'(x)\,d\mu(x).$$

If $J=I$, we get the result by applying the same argument to T^2. The second equality is also an immediate consequence of the Lemma.

(2) Let β, $\eta>0$ satisfy $\beta(x-1)\leqq\log x\leqq x-1$ for $1\leqq x\leqq 1+\eta$. Choose $\varepsilon>0$ such that $T'(x)\leqq 1+\eta$ for all $x\in B(j)$ with $|x-x_j|<\varepsilon$ and all $j\in J$. Putting

$$X_\varepsilon=\bigcup_{j\in J}\{x\in B(j):|x-x_j|<\varepsilon\}$$

we have

$$\boldsymbol{h}(T,\mu)=\int_{X_\varepsilon}\log T'(x)\,d\mu(x)+\int_{[0,1]\setminus X_\varepsilon}\log T'(x)\,d\mu(x).$$

By condition (4) imposed on T there exists a constant $c>0$ such that $(1/c)\lambda(B(i))\leqq\omega(i)\leqq c\lambda(B(i))$ a.e. for all $i\in I$. Therefore,

$$-\log c-\sum_{i\in I}\lambda(B(i))\log\lambda(B(i))\leqq\int_0^1\log T'(x)\,d\lambda(x)$$

$$\leqq\log c-\sum_{i\in I}\lambda(B(i))\log\lambda(B(i)).$$

Hence by assumption

$$\int_0^1\log T'(x)\,d\lambda(x)<\infty.$$

Since $d\mu/d\lambda$ is bounded above on $[0,1]\setminus X_\varepsilon$ this implies

$$\int_{[0,1]\setminus X_\varepsilon}\log T'(x)\,d\mu(x)<\infty.$$

Now,

$$\beta(T'(x)-1)\leqq\log T'(x)\leqq T'(x)-1 \qquad \text{for } x\in B(j)\cap X_\varepsilon,$$

hence

$$c_1\beta\int_{B(j)\cap X_\varepsilon} u_j(x)\,d\lambda(x) \leqq \int_{B(j)\cap X_\varepsilon} \log T'(x)\,d\mu(x)$$

$$\leqq c_2\int_{B(j)\cap X_\varepsilon} u_j(x)\,d\lambda(x),$$

where c_1, c_2 are the constants in (1.6). Taking into account that $u_j \in L_1(B(j),\lambda)$ if and only if

$$\int_{B(j)\cap X_\varepsilon} u_j(x)\,d\lambda(x) < \infty,$$

the assertion is proved.

(3) Let $B = B(b)$, $b \in I\setminus J$, and $S:[0,1]\to[0,1]$ be the first passage map with respect to B (see section 1, (iii)). As every S-cylinder is a T^*-cylinder, (1.4) implies that S has an ergodic invariant measure ν with density bounded away from zero and infinity. After a suitable normalisation of ν, (1.3) holds and therefore $\mu(B)=\nu([0,1])$. Hence the Lemma implies $\boldsymbol{h}(T,\mu)=\mu(B)h(S)$. Letting

$$j_n(x)=\sum_{k=0}^{n-1} 1_B(T^k x)$$

we have

$$B(\boldsymbol{k}_n(x)) = B(\tilde{k}_1,\cdots,\tilde{k}_{j_n(x)},*,\cdots,*),$$

where the blocks $\tilde{k}_j$ correspond to S-cylinders of order one, and the digits marked by $*$ are different from b. Thus,

$$-\log\nu(B(\tilde{k}_1,\cdots,\tilde{k}_{j_n(x)}))/j_n(x) \leqq -\log\nu(B(\boldsymbol{k}_n(x)))/j_n(x)$$

$$\leqq -((j_n(x)+1)/j_n(x))\log\nu(B(\tilde{k}_1,\cdots,\tilde{k}_{j_n(x)+1}))/(j_n(x)+1).$$

Taking into account that $\lim_{n\to\infty} j_n(x)=\infty$ a.e. we get, by applying McMillan's Theorem to S,

$$-\lim_{n\to\infty}\log\nu(B(\boldsymbol{k}_n(x)))\Big/\sum_{k=0}^{n-1}1_B(T^k x) = h(S) = \boldsymbol{h}(T,\mu)/\mu(B) \qquad \text{a.e.}$$

Since ν can be replaced by λ the result now follows from the Chacon–Ornstein Theorem.

If $J = I$, then applying the same argument to T^2 and $B = B(b_1,b_2)$, $b_1 \neq b_2$, we see that

$$-\lim_{n\to\infty} \log \lambda(B(\boldsymbol{k}_{2n}(x))) \Big/ \sum_{k=0}^{n-1} 1_B(T^{2k}x) = 2\boldsymbol{h}(T,\mu)/\mu(B) \qquad \text{a.e.}$$

Because

$$2\sum_{k=0}^{n-1} 1_B(T^{2k}x) \sim \sum_{k=0}^{2n-1} 1_B(T^k x) \qquad \text{a.e.}$$

the assertion is proved for the subsequence of even positive integers. The rest is a consequence of $B(\boldsymbol{k}_{2n+2}) \subseteq B(\boldsymbol{k}_{2n+1}) \subseteq B(\boldsymbol{k}_{2n})$. □

REMARKS. (1) From the proof of (2) it can be seen that $\boldsymbol{h}(T,\mu)$ is infinite if $-\Sigma_{i\in I}\lambda(B(i))\log\lambda(B(i))$ is infinite. If this sum is finite, it depends on the behaviour of T at the fixed points x_j, $j \in J$, whether $\boldsymbol{h}(T,\mu)$ is finite or infinite. A sufficient condition for u_j to belong to $L_1(B(j),\lambda)$ is, for example, that T is r times continuously differentiable in a neighbourhood of x_j, $r \geqq 2$, and $T^{(i)}(x_j) \neq 0$ for some $i \in \{2,\cdots,r\}$. Intuitively speaking, this means that points near to x_j do not move too slowly under iteration of T, or equivalently, that the wandering rate of T is not too large. Otherwise, as the following example illustrates, $\boldsymbol{h}(T,\mu)$ is infinite.

Let $f(0)=0$, $f(x)=x+x^2e^{-1/x}$, $x>0$, and let $a\in(0,1)$ be determined by $f(a)=1$. Define $T:[0,1]\to[0,1]$ by

$$T(x)=\begin{cases} f(x), & x\in B(0)=[0,a], \\ (x-a)/(1-a), & x\in B(1)=(a,1]. \end{cases}$$

Then $u_0(x)=2+1/x \notin L_1(B(0),\lambda)$, hence $\boldsymbol{h}(T,\mu)=\infty$. Note that the invariant density of T has an essential singularity at 0. In fact,

$$\frac{d\mu}{d\lambda}(x)=g(x)e^{1/x}/x, \qquad g \text{ continuous and positive on } [0,1].$$

Furthermore, by the lemmas of section 3, $w_n(T)=g(0)n/\log n$.

(2) If T has finite entropy, we define (adopting an idea from [1])

$$\hat{w}_n(T)=w_n(T)/\boldsymbol{h}(T,\mu).$$

Then, if T_1, T_2 are weakly isomorphic transformations with finite entropy,

$$\hat{w}_n(T_1)=\hat{w}_n(T_2) \qquad \text{holds true.}$$

Clearly, $\{\hat{w}_n(T)\}$ is a still more powerful invariant.

Consider, for example, the following two transformations:

$$T_1(x)=\begin{cases} x/(1-x), & 0\leqq x\leqq \frac{1}{2}, \\ 1/x-1, & \frac{1}{2}\leqq x\leqq 1, \end{cases} \qquad \frac{d\mu_1}{d\lambda}(x)=1/x;$$

$$T_2(x)=\begin{cases} x/(1-x), & 0\leqq x\leqq \frac{1}{2}, \\ 2-2x, & \frac{1}{2}\leqq x\leqq 1, \end{cases} \qquad \frac{d\mu_2}{d\lambda}(x)=2/x(2-x).$$

In both cases the wandering rate is equal to $\{\log n\}$.

Since T_1^* is the continued fraction transformation it follows that $\boldsymbol{h}(T_1,\mu_1)=\pi^2/6\approx 1.645$ (cf. Theorem 5). On the other hand,

$$\begin{aligned}\boldsymbol{h}(T_2,\mu_2)&=-4\int_0^{1/2}\frac{\log(1-x)}{x(2-x)}dx+2\log 2\int_{1/2}^1\frac{dx}{x(2-x)}\\ &=-4\int_{1/2}^1\frac{\log x}{1-x^2}dx+\log 2\cdot\log 3\\ &=\pi^2/2-\log 2\cdot\log 3-4\sum_{n=0}^{\infty}2^{-(2n+1)}(2n+1)^{-2}\\ &\approx 2.112,\end{aligned}$$

i.e. the 'normalized' wandering rates $\hat{w}_n(T_1)$ and $\hat{w}_n(T_2)$ are different. Thus the systems (T_1,μ_1) and (T_2,μ_2) are not weakly isomorphic.

(3) We conclude by calculating the entropy of $Tx=\tan x$, $x\in\mathbf{R}$. With the notations of Example 2 in section 3 we get

$$\begin{aligned}\boldsymbol{h}(T,\mu)&=\pi\int_0^1\log S'(x)\frac{dx}{\cos^2\pi x}\\ &=-2\pi\int_0^1\log\sin\pi x\frac{dx}{\cos^2\pi x}\\ &=2\pi\int_0^1 dx\\ &=2\pi,\end{aligned}$$

and

$$\hat{w}_n(T)=(1/\pi)\sqrt{2n/3}.$$

References

1. J. Aaronson, *Rational ergodicity and a metric invariant for Markov shifts*, Isr. J. Math. **27** (1977), 93–123.

2. J. Aaronson, *Ergodic theory for inner functions of the upper half plane*, Ann. Inst. Henri Poincaré, Sec. B, **14** (1978), 233–253.

3. J. Aaronson, *The asymptotic distributional behaviour of transformations preserving infinite measures*, J. Analyse Math. **39** (1981), 203–233.

4. R. L. Adler, *F-expansions revisited*, in *Recent Advances in Topological Dynamics*, Lecture Notes in Math., No. 318, Springer-Verlag, 1973, pp. 1–5.

5. R. Bowen, *Invariant measures for Markov maps of the interval*, Commun. Math. Phys. **69** (1979), 1–17.

6. P. Collet and J.-P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems*, Birkhäuser, Basel–Boston–Stuttgart, 1980.

7. S. Foguel and M. Lin, *Some ratio limit theorems for Markov operators*, Z. Wahrscheinlichkeitstheorie und Verw. Geb. **23** (1972), 55–66.

8. M. Halfant, *Analytic properties of Rényi's invariant density*, Isr. J. Math. **27** (1977), 1–20.

9. S. Kakutani, *Induced measure preserving transformations*, Proc. Imp. Acad. Sci. Tokyo **19** (1943), 635–641.

10. J. H. B. Kemperman, *The ergodic behaviour of a class of real transformations*, in *Stochastic Processes and Related Topics* (Proc. Summer Research Inst. on Statistical Inference for Stochastic Processes, Indiana University, Bloomington, 1974), Academic Press, New York, 1975, pp. 249–258.

11. U. Krengel, *Entropy of conservative transformations*, Z. Wahrscheinlichkeitstheorie und Verw. Geb. **7** (1967), 161–181.

12. Tien-Yien Li and F. Schweiger, *The generalized Boole's transformation is ergodic*, Manuscripta Math. **25** (1978), 161–167.

13. G. Pianigiani, *First return map and invariant measures*, Isr. J. Math. **35** (1980), 32–48.

14. A. Rényi, *Representations of real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hung. **8** (1957), 477–493.

15. V. A. Rohlin, *Exact endomorphisms of a Lebesgue space*, Am. Math. Soc. Tansl. II. Ser. **39** (1964), 1–36.

16. F. Schweiger, *Some remarks on ergodicity and invariant measures*, Michigan Math. J. **22** (1975), 308–318.

17. F. Schweiger, *Ergodic properties of fibered systems*, Proc. Sixth Conference on Probability Theory, Braşov, Romania, 1979, pp. 221–228.

18. F. Schweiger, *Invariant measures for piecewise linear fractional maps*, J. Austral. Math. Soc. (Ser. A) **34** (1983), 55–59.

19. F. Schweiger and M. Thaler, *Unimodal maps on* $[-1,1]$ *with the indifferent fixed point* $x = -1$, Arbeitsberichte 3/1982, Math. Institut der Univ. Salzburg, pp. 39–44.

20. M. Thaler, *Estimates of the invariant densities of endomorphisms with indifferent fixed points*, Isr. J. Math. **37** (1980), 303–314.

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